

The associativity rule in pathwise functional Itô calculus

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Abstract

In this paper we establish the associativity property of the pathwise Itô integral in a functional setting for continuous integrators. Here, associativity refers to the computation of the Itô differential of an Itô integral, by means of the intuitive cancellation of the Itô differential and integral signs. As an application, we derive existence and uniqueness results for linear Itô differential equations in a functional setting.

1 Introduction

Dupire [14] and Cont and Fournié [4, 6] have recently introduced a new type of stochastic calculus, known as *functional Itô calculus*. It is based on an extension of the classical Itô formula to functionals depending on the entire past evolution of the underlying path, and not only on its current value. The approach taken in [4] is a direct extension of the non-probabilistic Itô formula of Föllmer [15] to non-anticipative functionals on Skorohod space. These functionals are required to possess certain directional derivatives which may be computed pathwise, but no Fréchet differentiability is imposed. An alternative approach, which to some extent still relies on probabilistic arguments, was introduced by Cosso and Russo [7]; it is based on the theory of stochastic calculus via regularization [20, 12, 9, 10, 13, 11].

In recent years, pathwise Itô calculus has been particularly popular in mathematical finance and economics; see, e.g., [2, 3, 8, 16, 17, 21, 25]. This is due to the fact that the results derived with the help of the pathwise Itô calculus are robust with respect to model risk that might stem from a misspecification of probabilistic dynamics. In this sense, there is also a close link to robust statistics. The only assumption on the underlying paths is that they admit the quadratic variation in the sense of [15].

Our first contribution in this paper is a slight extension of the functional change of variables formula from [4], which is motivated by the fact that functionals of interest often depend on additional

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arguments such as quadratic variation, moving average, or running maximum of the underlying path. These quantities, however, are not regular enough to fit fully into the framework of [4] (see also the discussions in [6] and [14]). To this end, we will allow our functionals F to depend on an additional variable A that corresponds to a general path of bounded variation, and then extend the notions of the horizontal and vertical derivatives to functionals of this type. This extension will be crucial for the proof of our associativity rule.

Let ξ be an admissible integrand for a continuous path X that satisfies the assumptions of Föllmer's pathwise Itô formula and let η an admissible integrand for the path $Y(t) := \int_0^t \xi(s) dX(s)$. Then our main result states that $\eta\xi$ is again an admissible integrand for X , and we have the intuitive cancellation property

$$\int_0^T \eta(s) dY(s) = \int_0^T \eta(s)\xi(s) dX(s).$$

This cancellation rule is often called the *associativity* of the integral. Note that in standard stochastic calculus it follows immediately from an application of the Kunita–Watanabe characterization of the stochastic integral. In our present pathwise setting, however, this characterization is not available, and the proof of the associativity property becomes surprisingly involved, as only analytical tools are at our disposition.

Nevertheless, just as in standard stochastic calculus, associativity is a fundamental property of the Itô integral and crucial for many applications. For instance, in [22], a basic version of the associativity rule was derived so as to give a pathwise treatment of constant-proportion portfolio insurance strategies (CPPI). Our original motivation for deriving an associativity rule within functional Itô calculus stems from the fact that it is crucial for analyzing functionally dependent strategies in a pathwise version of stochastic portfolio theory; see our companion paper [24]. To illustrate already in our present paper why the associativity rule is such a crucial property, we use it in Section 4 to prove existence and uniqueness results for pathwise linear Itô differential equations whose coefficients are non-anticipative functionals.

The paper is organized as follows. In Section 2 we recall from [4] the basic notions of functional Itô calculus and provide our slightly extended change of variables formula for non-anticipative functionals depending on an additional bounded variation component. With this at hand, we can state and show in Section 3 the associativity of the pathwise Itô integral $\int \xi(s) dX(s)$. Our applications to pathwise Itô differential equations are given in Section 4.

2 Preliminaries

2.1 Non-anticipative functionals and functional derivatives on spaces of paths

In the following we will first describe our framework, slightly extending the definitions and notations introduced in [4] and [14]. In the second step, we will derive our slightly extended functional change of variables formula for a continuous path X . However, as the definition of functional derivatives requires us to apply discontinuous shocks even in case X is continuous, we still have to consider functionals defined on the space of càdlàg paths.

For the sake of simplicity, we keep our notation as close as possible to the one in [4]. More precisely, let $T > 0$ and $D \subset \mathbb{R}^n$ be an arbitrary subset of \mathbb{R}^n . As in [4], a “ D -valued càdlàg function” is a right-

continuous function $f : [0, T] \mapsto D$ with left limits such that for each $t \in (0, T]$, $f(t-) \in D$. By $\Delta f(t) := f(t) - f(t-)$ we denote the jump of f at time t . By $f^t = (f(u \wedge t))_{0 \leq u \leq T}$ we denote the stopped path at t . As in [4], we use curly letters to denote the class of càdlàg functions with values in a certain set. More precisely, we write $\mathcal{D}^T = D([0, T], D)$ for the space of D -valued càdlàg functions on $[0, T]$, and $\mathcal{D}^t \subset \mathcal{D}^T$ for the space of D -valued càdlàg paths stopped at time t . Analogously, $\mathcal{D}_I^t \subset \mathcal{D}_I$ is the set of D -valued càdlàg paths on a subinterval $I \subset [0, T]$, stopped at time t . By $C(I, D)$ we denote the set of continuous functions on I with values in D . Note that we work with *stopped* paths instead of restrictions as in [4]. This is simpler and clearly equivalent to working with restricted paths (see also [6]). In the sequel, let $U \subset \mathbb{R}^d$ be an open subset of \mathbb{R}^d and $S \subset \mathbb{R}^m$ a Borel subset of \mathbb{R}^m .

Definition 2.1. A family $F : [0, T] \times \mathcal{U}^T \times \mathcal{S}^T \mapsto \mathbb{R}$ of functionals is said to be *non-anticipative* if

$$\forall(t, X, A) \in [0, T] \times \mathcal{U}^T \times \mathcal{S}^T, \quad F(t, X, A) = F(t, X^t, A^t). \quad (1)$$

In contrast to [4], we will require throughout this paper that the function A in (1) has components of bounded variation. In this way, we will be able to deal with non-anticipative functionals F that depend on quantities such as the running maximum $\max_{u \leq t} X(u)$ or the quadratic variation $[X]$ of a trajectory X , which are often not absolutely continuous in t . Functionals depending on such quantities are not covered by the Itô formula from [4]. The ability to deal with such functionals, however, is beneficial in view of many applications in mathematical finance. It will also turn out to be very natural and convenient in our proof of the associativity rule. Note that in the setting of [4] an additional argument of F , denoted V , is admitted. Its role, however, is rather limited due to the requirements on the horizontal derivative, which in [4] are more restrictive than here.

If I is a time interval, the class of right-continuous functions $A : I \mapsto \mathbb{R}^m$ whose components are of bounded variation will be denoted by $BV^m(I)$. The subset of continuous functions in $BV^m(I)$ will be denoted by $CBV^m(I)$. We will also use the notation $\mathcal{D}_{I,BV}^t := \mathcal{D}_I^t \cap BV^m(I)$ and $\mathcal{D}_{I,CBV}^t := \mathcal{D}_I^t \cap CBV^m(I)$.

For the definition of functional derivatives, we need to introduce the following notion.

Definition 2.2 (Perturbation on path spaces). Let $X \in D([0, T], U)$ and X^t be the stopped path at t . For $h \in \mathbb{R}^d$ sufficiently small, the *vertical perturbation* $X^{t,h}$ of the stopped path X^t is defined as the càdlàg path obtained by shifting the value at t by the quantity h :

$$X^{t,h}(u) = X^t(u), \quad u \in [0, t), \quad X^{t,h}(u) = X(t) + h, \quad u \in [t, T], \quad (2)$$

or, equivalently, $X^{t,h}(u) = X^t(u) + h \mathbb{1}_{[t,T]}(u)$.

Since we work with stopped instead of restricted paths, we can use the standard supremum norm on path space:

$$d_\infty((X, A), (X', A')) = \sup_{u \in [0, T]} |X(u) - X'(u)| + \sup_{u \in [0, T]} |A(u) - A'(u)| \quad (3)$$

for $(X, A), (X', A') \in \mathcal{U}^T \times \mathcal{S}^T$.

Definition 2.3 (Regularity properties). (i) A non-anticipative functional F is said to be *left-continuous* (notation: $F \in \mathbb{F}_l^\infty$) if

$$\forall t \in [0, T], \forall \epsilon > 0, \forall (X, A) \in \mathcal{U}^t \times \mathcal{S}_{BV}^t,$$

$$\begin{aligned} & \exists \eta > 0 \text{ such that } \forall h \in [0, t], \forall (X', A') \in \mathcal{U}^{t-h} \times \mathcal{S}_{BV}^{t-h}, \\ & d_\infty((X, A), (X', A')) + h < \eta \Rightarrow |F(t, X, A) - F((t-h), X', A')| < \epsilon. \end{aligned} \quad (4)$$

(ii) A non-anticipative functional F is said to be *boundedness-preserving* (notation: $F \in \mathbb{B}$) if for any compact subset $K \subset U$ there exists a constant C_K such that

$$\forall t \in [0, T], \forall (X, A) \in \mathcal{K}^t \times \mathcal{S}_{BV}^t, \quad |F(t, X, A)| < C_K. \quad (5)$$

Note that if $F \in \mathbb{B}$, then it is *locally bounded* in the neighborhood of any given path. That is,

$$\begin{aligned} & \forall (X, A) \in \mathcal{U}^T \times \mathcal{S}_{BV}^T, \exists C > 0, \eta > 0 \text{ such that} \\ & \forall t \in [0, T], \forall (X', A') \in \mathcal{U}^t \times \mathcal{S}_{BV}^t, \\ & d_\infty((X^t, A^t), (X', A')) < \eta \Rightarrow \forall t \in [0, T], \quad |F(t, X', A')| \leq C. \end{aligned} \quad (6)$$

We next introduce our notion of *horizontal derivative* (with respect to some measure), which is motivated by the desire to lessen smoothness assumptions on those functionals, for which a change of variables formula can be derived. This extends the horizontal derivative from [4] and [7].

Definition 2.4 (Horizontal derivative). Let F be a non-anticipative functional and $(X, A) \in \mathcal{U}^T \times \mathcal{S}_{BV}^T$. Since the components of A are functions of bounded variation and, hence, correspond to finite measures μ_k , $k = 1, \dots, m$, on $[0, T]$, we can introduce the vector-valued measure μ on $[0, T]$ via $\mu(ds) := (ds, A_1(ds), \dots, A_m(ds))^\top$. The *horizontal derivative* of F at (t, X^t, A^t) (with respect to μ) is defined as the vector

$$\mathcal{D}F(t, X^t, A^t) = (\mathcal{D}_0 F(t, X^t, A^t), \mathcal{D}_1 F(t, X^t, A^t), \dots, \mathcal{D}_m F(t, X^t, A^t))^\top,$$

where

$$\mathcal{D}_0 F(t, X^t, A^t) := \lim_{h \rightarrow 0^+} \frac{F(t, X^{t-h}, A^{t-h}) - F(t-h, X^{t-h}, A^{t-h})}{h} \quad (7)$$

$$\mathcal{D}_k F(t, X^t, A^t) := \lim_{h \rightarrow 0^+} \frac{F(t, X^{t-h}, A_1^{t-h}, \dots, A_k^t, \dots, A_m^{t-h}) - F(t, X^{t-h}, A^{t-h})}{\mu_k((t-h, t])} \quad (8)$$

for $k = 1, \dots, m$, if the corresponding limits exist. In addition, it will be convenient to set $\mathcal{D}F(t, X^t, A^t) = 0$ for $t = 0$.

If (7) and (8) are well-defined for all (X, A) , then the map

$$\begin{aligned} \mathcal{D}F : [0, T] \times \mathcal{U}^T \times \mathcal{S}_{BV}^T &\mapsto \mathbb{R}^{m+1} \\ (t, X, A) &\mapsto \mathcal{D}F(t, X^t, A^t) \end{aligned} \quad (9)$$

defines a non-anticipative vector-valued functional $\mathcal{D}F : [0, T] \times \mathcal{U}^T \times \mathcal{S}_{BV}^T \mapsto \mathbb{R}^{m+1}$, the extended *horizontal derivative* of F .

Note that in contrast to the definition of the horizontal derivative in [4] as time derivative, the definition (7) is based on a limit on the left, and thus only the past evolution of the underlying path is taken into account while no assumptions on the possible future values are made whatsoever. This modification is inspired by [7, Definition 2.9].

Definition 2.5 (Vertical derivative). A non-anticipative functional F is said to be *vertically differentiable with respect to X* at $(X^t, A^t) \in \mathcal{U}^t \times \mathcal{S}_{BV}^t$ if the map $\mathbb{R}^d \ni e \rightarrow F(t, X^{t,e}, A^t)$ is differentiable at 0. Its gradient at 0,

$$\begin{aligned} \nabla_X F(t, X^t, A^t) &= (\partial_i F(t, X^t, A^t), i = 1, \dots, d), \text{ where} \\ \partial_i F(t, X^t, A^t) &= \lim_{h \rightarrow 0} \frac{F(t, X^{t, h e_i}, A^t) - F(t, X^t, A^t)}{h}, \end{aligned} \quad (10)$$

is called the *vertical derivative* of F at (t, X^t, A^t) , with respect to X . If (10) is well-defined for all (X, A) , the map

$$\begin{aligned} \nabla_X F : [0, T] \times \mathcal{U}^T \times \mathcal{S}_{BV}^T &\mapsto \mathbb{R}^d \\ (t, X, A) &\rightarrow \nabla_X F(t, X^t, A^t) \end{aligned} \quad (11)$$

defines a non-anticipative functional $\nabla_X F$ with values in \mathbb{R}^d , which we call the *vertical derivative of F with respect to X* .

Example 2.1. Let $X \in \mathcal{U}^T$, $A \in \mathcal{S}_{BV}^T$ in the following. If $F(t, X^t, A^t) = f(X(t), A(t))$ for a smooth function $f(a, x)$, then $\mathcal{D}F(t, X^t, A^t) = \nabla_a f(A(t), X(t))$ and $\nabla_X F(t, X^t, A^t) = \nabla_x f(A(t), X(t))$, where $\nabla_a f(A(t), X(t))$ is the gradient of the function $f(a, x)$ with respect to the first argument, evaluated at $(A(t), X(t))$, and, analogously, $\nabla_x f(A(t), X(t))$ is the gradient of the function $f(a, x)$ with respect to the second argument, evaluated at $(A(t), X(t))$.

Example 2.2. Sometimes, a quantity of interest can either be considered as a path-dependent functional of $X \in \mathcal{U}^T$ only or as an additional trajectory of bounded variation. The latter possibility allows us to include functionals that may not be regular enough for the setting of [4] or [14]. This illustrates one advantage of our extended approach (see also the paragraph following Definition 2.1.).

- (i) Consider the time average of the first component, $F(t, X^t) = \int_0^t X_1(s) ds$. Alternatively, it can be represented as $A_X(t)$, where $A_X(t) := \int_0^t X_1(s) ds$, because the time average is a function of bounded variation. In the first approach, we have $\mathcal{D}F(t, X^t) = X_1(t-)$ and $\partial_X F(t, X^t) = 0$. In the second approach, we have $\mathcal{D}A_X(t) = f'(A_X(t)) = 1$ with $f(a) = a$. Clearly, $\partial_X A_X(t) = 0$.
- (ii) Consider the functional $F(t, X^t) = [X_1](t)$, where $[X_1]$ is the pathwise quadratic variation of the first component in the sense of Föllmer [15] (see Definition 2.7). Alternatively, it can be represented as $A_X(t) := [X_1](t)$. In the first approach, we have $\mathcal{D}F(t, X^t) = 0$ and $\partial_X F(t, X^t) = 2(X_1(t) - X_1(t-))$. In the second approach, we have $\mathcal{D}A_X(t) = 1$ and $\partial_X A_X(t) = 0$.
- (iii) Consider the running maximum of the first component $F(t, X^t) = \max_{0 \leq s \leq t} X_1(s)$. Alternatively, it can be represented as $A_X(t) := \max_{0 \leq s \leq t} X_1(s)$, since the running maximum is a function of bounded variation. Then, F is not (vertically) differentiable in the first approach, and we would have to resort to smoothing techniques [14]. In the second approach, however, the horizontal derivative with respect to the measure corresponding to A_X does exist, and we have $\mathcal{D}A_X(t) = 1$.

If the functional F admits horizontal and vertical derivatives $\mathcal{D}F$ and $\nabla_X F$, we may iterate the above operations in order to define higher order horizontal and vertical derivatives.

Definition 2.6. Let $I \subset [0, T]$ be a subinterval of $[0, T]$ with nonempty interior, $\overset{\circ}{I}$. We denote by $\mathbb{C}^{j,k}(I)$ the set of all non-anticipative functionals F on $\cup_{t \in I} \mathcal{U}_I^t \times \mathcal{S}_{I,BV}^t$ such that:

- F is continuous at fixed times t , locally uniformly in t . That is,

$$\begin{aligned} \forall t \in [0, T], \forall \epsilon > 0, \forall (X, A) \in \mathcal{U}_I^t \times \mathcal{S}_{I,BV}^t, \\ \exists \eta > 0 \text{ such that } \forall (X', A') \in \mathcal{U}_I^{t'} \times \mathcal{S}_{I,BV}^{t'}, \\ d_\infty((X, A), (X', A')) + |t - t'| < \eta \Rightarrow |F(t', X, A) - F(t', X', A')| < \epsilon. \end{aligned} \quad (12)$$

- F admits j horizontal derivatives and k vertical derivatives with respect to X at all $(X, A) \in \mathcal{U}_I^t \times \mathcal{S}_{I,BV}^t$, $t \in I$.
- $\mathcal{D}^l F$, $l \leq j$, $\nabla_X^n F$, $n \leq k$, are left-continuous on I .

We recall now the notion of quadratic variation in the sense of Föllmer [15].

Definition 2.7 (Quadratic variation). Let $T > 0$, $(\mathbb{T}_n) = (\mathbb{T}_n)_{n \in \mathbb{N}}$ be a refining sequence of partitions of $[0, T]$, and $X, Y \in C([0, T], \mathbb{R})$. We denote by t' the successor of $t \in \mathbb{T}_n$, i.e.,

$$t' = \begin{cases} \min\{u \in \mathbb{T}_n \mid u > t\} & \text{if } t < T, \\ T & \text{if } t = T. \end{cases}$$

We say that X and Y admit the *continuous covariation* $[X, Y]$ along $(\mathbb{T}_n)_{n \in \mathbb{N}}$ if and only if for all $t \in [0, T]$ the sequence

$$\sum_{\substack{s \in \mathbb{T}_n \\ s \leq t}} (X(s') - X(s))(Y(s') - Y(s)) \quad (13)$$

converges to a finite limit, denoted $[X, Y](t)$, and if $t \mapsto [X, Y](t)$ is continuous. If $X = Y$, we say that X admits the *continuous quadratic variation* $[X]$ along $(\mathbb{T}_n)_{n \in \mathbb{N}}$ (notation: $X \in QV$), and we set $[X] := [X, X]$. We say that $X \in C([0, T], \mathbb{R}^d)$ admits the *continuous quadratic variation* along (\mathbb{T}_n) (notation: $X \in QV^d$) if and only if the functions X_i , $i = 1, \dots, d$, and $X_i + X_j$, $i, j = 1, \dots, d$, do. Writing \mathcal{S}_+^d for the class of symmetric nonnegative definite $d \times d$ matrices, the quadratic variation of $X \in C([0, T], \mathbb{R}^d)$ is given by the \mathcal{S}_+^d -valued function $[X]$, defined by

$$[X]_{ii} = [X_i], \quad [X]_{ij} = \frac{1}{2} \left([X_i + X_j] - [X_i] - [X_j] \right) = [X_i, X_j], \quad i \neq j. \quad (14)$$

Note that the quadratic variation depends strongly on the particular choice of the refining sequence of partitions. For example, it is shown in [18, p.47] that for any continuous function $X : [0, 1] \mapsto \mathbb{R}$ there exists a refining sequence of partitions along which the quadratic variation of X is identically zero. See also [23, Proposition 2.7] for an example where $[X, Y]$ does not exist even though both $[X]$ and $[Y]$ exist.

2.2 Change of variables formula

To prove our associativity result we will need the following change of variables formula, which is a slight extension of the corresponding formulae from [4] and [14]. Although its proof requires only minor adjustments to the one given in [4], it will be efficient to give it here, as we will need its approximation and convergence arguments in the proof of our associativity rule, Theorem 3.1. From now on we fix a refining sequence of partitions (\mathbb{T}_n) of $[0, T]$.

Theorem 2.1. *Let $(X, A) \in QV^d \cap \mathcal{U}^T \times \mathcal{S}_{CBV}^T$ and denote*

$$X^n(t) := \sum_{s \in \mathbb{T}_n} X(s') \mathbb{1}_{[s, s')}(t) + X(T) I_{\{T\}}(t), \quad 0 \leq t \leq T, \quad (15)$$

$$A^n(t) := \sum_{s \in \mathbb{T}_n} A(s) \mathbb{1}_{[s, s')}(t) + A(T) I_{\{T\}}(t), \quad 0 \leq t \leq T, \quad (16)$$

$$h_s^n := s' - s, \quad s, s' \in \mathbb{T}_n. \quad (17)$$

Suppose moreover that F is a left-continuous non-anticipative functional of class $\mathbb{C}^{1,2}([0, T])$ such that $\mathcal{D}F, \nabla_X F, \nabla_X^2 F \in \mathbb{B}$. Denote $X^{n, s-}$ the n -th approximation of X stopped at time $s-$. Then the pathwise Itô integral along (\mathbb{T}_n) , defined as

$$\int_0^T \nabla_X F(s, X^s, A^s) dX(s) := \lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \nabla_X F(s, X^{n, s-}, A^{n, s}) \cdot (X(s') - X(s)), \quad (18)$$

exists and

$$\begin{aligned} F(T, X^T, A^T) - F(0, X^0, A^0) &= \int_0^T \mathcal{D}F(s, X^s, A^s) \mu(ds) + \frac{1}{2} \sum_{i,j=1}^d \int_0^T \nabla_{X,ij}^2 F(s, X^s, A^s) d[X]_{ij}(s) \\ &\quad + \int_0^T \nabla_X F(s, X^s, A^s) dX(s). \end{aligned} \quad (19)$$

Proof. The proof uses the same Föllmer-type discretization techniques as are used in the proof of [4, Theorem 3]. Denote $\delta X_s^n = X(s') - X(s)$ for $s, s' \in \mathbb{T}_n$ and $|\mathbb{T}_n| := \sup_{s \in \mathbb{T}_n} |s' - s|$ the mesh of \mathbb{T}_n . Since X and A are continuous, and hence uniformly continuous on $[0, T]$, it follows that

$$\eta_n := \sup_{\alpha, \beta \in [0, T], |\alpha - \beta| \leq |\mathbb{T}_n|} (|A(\alpha) - A(\beta)| + |X(\alpha) - X(\beta)|) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (20)$$

Since $\nabla_X^2 F, \mathcal{D}F \in \mathbb{B}$, for n large enough there exists a constant $C > 0$ such that

$$\begin{aligned} \forall t < T, \forall (X', A') \in \mathcal{U}^t \times \mathcal{S}_{BV}^t, \\ d_\infty((X^t, A^t), (X', A')) < \eta_n \quad \Rightarrow \quad |\mathcal{D}F(t, X', A')| \leq C, \quad \|\nabla_X^2 F(t, X', A')\| \leq C. \end{aligned}$$

For $s \in \mathbb{T}_n$, consider the following decomposition of increments into “horizontal” and “vertical” terms:

$$\begin{aligned} F(s', X^{n, s'-}, A^{n, s'}) - F(s, X^{n, s-}, A^{n, s}) &= F(s', X^{n, s'-}, A^{n, s'}) - F(s, X^{n, s}, A^{n, s}) \\ &\quad + F(s, X^{n, s}, A^{n, s}) - F(s, X^{n, s-}, A^{n, s}). \end{aligned} \quad (21)$$

The first term on the right-hand side of (21) is equal to $\psi(h_s^n) - \psi(0)$, where $\psi(u) := F(s+u, X^{n,s}, A^{n,s+u})$. Since F admits a horizontal derivative in the sense of (7), (8), we can write

$$\begin{aligned} F(s', X^{n,s'-}, A^{n,s'}) - F(s, X^{n,s}, A^{n,s}) &= \int_{(0,s'-s]} \mathcal{D}F(s+u, X^{n,s}, A^{n,s+u-}) \mu_n(du) \\ &= \int_{(s,s']} \mathcal{D}F(u, X^{n,s}, A^{n,u-}) \mu_n(du), \end{aligned} \quad (22)$$

where μ_n is the measure corresponding to the n -th approximation A^n of A , i.e.,

$$\mu_n(ds) = (ds, A_{n,1}(ds), \dots, A_{n,m}(ds))^\top.$$

For the second term on the right-hand side of (21), we have

$$F(s, X^{n,s}, A^{n,s}) - F(s, X^{n,s-}, A^{n,s}) = \phi(\delta X_s^n) - \phi(0), \quad (23)$$

where $\phi(u) = F(s, X^{n,s-,u}, A^{n,s})$. Since $F \in \mathbb{C}^{1,2}([0, T])$, the function ϕ is well-defined and twice continuously differentiable in the neighborhood $\mathcal{B}(X(s), \eta_n) \subset U$, with

$$\nabla \phi(u) = \nabla_X F(s, X^{n,s-,u}, A^{n,s}), \quad (24)$$

$$\nabla^2 \phi(u) = \nabla_X^2 F(s, X^{n,s-,u}, A^{n,s}). \quad (25)$$

Hence, a second-order Taylor expansion of ϕ at $u = 0$ yields

$$\begin{aligned} F(s, X^{n,s}, A^{n,s}) - F(s, X^{n,s-}, A^{n,s}) &= \nabla_X F(s, X^{n,s-}, A^{n,s}) \delta X_s^n \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \nabla_{X,ij}^2 F(s, X^{n,s-}, A^{n,s}) \delta X_{i,s}^n \delta X_{j,s}^n + r_s^n, \end{aligned} \quad (26)$$

where for some $\theta \in [0, 1]$,

$$r_s^n := \frac{1}{2} \sum_{i,j=1}^d \left(\nabla_{X,ij}^2 F(s, X^{n,s-, \theta \delta X_s^n}, A^{n,s}) - \nabla_{X,ij}^2 F(s, X^{n,s-}, A^{n,s}) \right) \delta X_{i,s}^n \delta X_{j,s}^n.$$

We now sum over $s \in \mathbb{T}_n$.

- The left-hand side of (21) sums up to $F(T, X^{n,T-}, A^{n,T}) - F(0, X^0, A^0)$, which converges to $F(T, X^{T-}, A^T) - F(0, X^0, A^0)$, by left-continuity of F . Since X and A are continuous, this is equal to $F(T, X^T, A^T) - F(0, X^0, A^0)$.
- For the first term on the right-hand side of (21), we have

$$\sum_{s \in \mathbb{T}_n} F(s', X^{n,s'-}, A^{n,s'}) - F(s, X^{n,s}, A^{n,s}) = \int_{(0,T]} \mathcal{D}F(u, X^{n,s(u)}, A^{n,u-}) \mu_n(du), \quad (27)$$

in conjunction with (22). Here, we set $s(u) := s$ such that $u \in [s, s')$, $s, s' \in \mathbb{T}_n$. The integrand converges to $\mathcal{D}F(u, X^u, A^u)$, is bounded by C , and both are left-continuous in u by [4, Proposition 1]. Moreover, the sequence of finite measures μ_n , corresponding to the approximations A_n

of A , converges vaguely to the atomless measure μ , corresponding to A , so we can use a “diagonal lemma” for vague convergence of measures in the form of [4, Lemma 12] to obtain that (27) converges to

$$\int_{(0,T]} \mathcal{D}F(u, X^u, A^u) \mu(\mathrm{d}u) = \int_0^T \mathcal{D}F(s, X^s, A^s) \mu(\mathrm{d}s). \quad (28)$$

- For the second term on the right-hand side of (21), we have

$$\begin{aligned} \sum_{s \in \mathbb{T}_n} F(s, X^{n,s}, A^{n,s}) - F(s, X^{n,s-}, A^{n,s}) &= \sum_{s \in \mathbb{T}_n} \nabla_X F(s, X^{n,s-}, A^{n,s}) \cdot \delta X_s^n \\ &+ \frac{1}{2} \sum_{i,j=1}^d \sum_{s \in \mathbb{T}_n} \nabla_{X,ij}^2 F(s, X^{n,s-}, A^{n,s}) \delta X_{i,s}^n \delta X_{j,s}^n + \sum_{s \in \mathbb{T}_n} r_s^n. \end{aligned} \quad (29)$$

The quantity $\nabla_X^2 F(s, X^{n,s-}, A^{n,s}) \mathbb{1}_{u \in (s, s']}$ is bounded by C , converges to $\nabla_X^2 F(u, X^u, A^u)$, by left-continuity of $\nabla_X^2 F$, and both are left-continuous in u , by [4, Proposition 1]. Since $X \in QV^d$, we can again apply [4, Lemma 12] to infer that

$$\frac{1}{2} \sum_{i,j=1}^d \sum_{s \in \mathbb{T}_n} \nabla_{X,ij}^2 F(s, X^{n,s-}, A^{n,s}) \delta X_{i,s}^n \delta X_{j,s}^n \rightarrow \frac{1}{2} \sum_{i,j=1}^d \int_0^T \nabla_{X,ij}^2 F(s, X^s, A^s) \mathrm{d}[X]_{ij}(s). \quad (30)$$

Using the same argument, since $|r_s^n|$ is bounded by $\epsilon_s^n |\delta X_s^n|^2$, where ϵ_s^n converges to 0, the remainder term, $\sum_{s \in \mathbb{T}_n} r_s^n$, converges to 0.

Since all terms considered converge, the limit

$$\lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \nabla_X F(s, X^{n,s-}, A^{n,s}) \cdot (X(s') - X(s))$$

exists, and the theorem follows. \square

Lemma 2.1. *Suppose that F satisfies the regularity assumptions from Theorem 2.1. Then for any $(X, A) \in C([0, T], U) \times \mathcal{S}_{CBV}^T$ the map $t \mapsto F(t, X^t, A^t)$ is continuous.*

Proof. We first show the left-continuity of $t \mapsto F(t, X^t, A^t)$. Since X and A are continuous, and hence uniformly continuous on $[0, T]$, we have that for $h > 0$ sufficiently small,

$$d_\infty((X^{t-h}, A^{t-h}), (X^t, A^t)) = \sup_{u \in [t-h, t]} |X(u) - X(t-h)| + \sup_{u \in [t-h, t]} |A(u) - A(t-h)| \rightarrow 0 \text{ as } h \rightarrow 0^+.$$

Thus, the left-continuity of F implies that $F(t-h, X^{t-h}, A^{t-h}) - F(t, X^t, A^t) \rightarrow 0$ as $h \rightarrow 0^+$. Analogously, since X and A are continuous, and hence uniformly continuous on $[0, T]$, we have that for $h > 0$ sufficiently small,

$$d_\infty((X^{t+h}, A^{t+h}), (X^t, A^t)) = \sup_{u \in [t, t+h]} |X(u) - X(t)| + \sup_{u \in [t, t+h]} |A(u) - A(t)| \rightarrow 0 \text{ as } h \rightarrow 0^+.$$

This implies that

$$F(t+h, X^{t+h}, A^{t+h}) - F(t+h, X^t, A^t) \rightarrow 0 \text{ as } h \rightarrow 0^+, \quad (31)$$

due to the continuity at fixed times, which holds locally uniformly. Moreover, since X and A are continuous and the horizontal derivative is boundedness-preserving, we infer that

$$F(t+h, X^t, A^t) - F(t, X^t, A^t) = \int_t^{t+h} \mathcal{D}_0 F(u, X^u, A^u) du \rightarrow 0 \text{ as } h \rightarrow 0^+. \quad (32)$$

Using (31) and (32) yields

$$\begin{aligned} F(t+h, X^{t+h}, A^{t+h}) - F(t, X^t, A^t) &= F(t+h, X^{t+h}, A^{t+h}) - F(t+h, X^t, A^t) \\ &\quad + F(t+h, X^t, A^t) - F(t, X^t, A^t) \rightarrow 0 \text{ as } h \rightarrow 0^+, \end{aligned}$$

which shows the right-continuity. \square

Since the first and second integrals in (19) are Riemann-Stieltjes integrals, and hence continuous as functions in t , using Lemma 2.1 gives the following result.

Corollary 2.1. *The pathwise Itô integral defined in (18), as a function in t , is continuous.*

The limit in (18) depends on the choice of the refining sequence of partitions (\mathbb{T}_n) (see [4, Remark 5]), but, as opposed to [4], we do not make this dependence explicit in the notation to keep things simple. Theorem 2.1 implies in particular that the pathwise Itô integral is well-defined for integrands that are vertical derivatives of non-anticipative $\mathbb{C}^{1,2}$ -functionals satisfying the regularity conditions from Theorem 2.1. This allows us to formalize the notion of a locally admissible integrand, as follows.

Definition 2.8. Let $X \in QV^d \cap \mathcal{U}^T$. A function $t \mapsto \xi(t) \in \mathbb{R}^d$ is called a *locally admissible integrand* for X if there exists a partition $\mathbb{T} = \{t_0, \dots, t_N\}$ of $[0, T]$ such that for all $k = 1, \dots, N$ there are $m_k \in \mathbb{N}$, $A_k \in \mathcal{S}_{[t_{k-1}, t_k], CBV}$, and F_k as in Theorem 2.1 so that

$$\xi(t) = \sum_{k=1}^N \nabla_X F_k(t, X_k^t, A_k^t) \mathbb{1}_{t \in [t_{k-1}, t_k]}.$$

Here, $X_k := X|_{[t_{k-1}, t_k]}$ is the restriction of X to $[t_{k-1}, t_k]$, and we require

$$F_k(t_k, X_k^{t_k}, A_k^{t_k}) = F_{k+1}(t_k, X_k^{t_k}, A_k^{t_k}), \quad k = 1, \dots, N-1. \quad (33)$$

Remark 2.1. Let us point out that the above definition allows for a large class of integrands, which include generalized delta hedging strategies for many exotic and plain-vanilla options in complete market models such as local volatility models that are relevant for practical applications. It thus extends [22, Remark 4] to the functional setting. Moreover, A in the above definition can for instance be a continuous function of the running average of X , i.e., $t \mapsto \int_{(t-\delta)_+}^t X(s) ds$, or its running maximum, $t \mapsto \max_{(t-\delta)_+ \leq s \leq t} X(s)$, since these functions are of bounded variation on $[0, T]$.

Remark 2.2. Also note that the above definition of a locally admissible integrand can be equivalently written as follows: A function $t \mapsto \xi(t) \in \mathbb{R}^d$ is called a *locally admissible integrand* for $X \in QV^d \cap \mathcal{U}^T$ if for every $t \in [0, T]$ there exists $\epsilon > 0$ such that there are $m_\epsilon \in \mathbb{N}$, a continuous function A_ϵ of bounded variation on $[t-\epsilon, t+\epsilon]$, and F_ϵ as in Theorem 2.1 such that

$$\xi(t) = \nabla_X F_\epsilon(t, X_\epsilon^t, A_\epsilon^t) \mathbb{1}_{t \in [t-\epsilon, t+\epsilon]}.$$

Here, $X_\epsilon := X|_{[t-\epsilon, t+\epsilon]}$ is the restriction of X to $[t-\epsilon, t+\epsilon]$.

The following result extends the covariation formula for the pathwise (classical) Itô integral from [25, 22]. Also note that when finishing the submission of our paper, we became aware of the preprint [1], where a pathwise isometry formula is also derived, but our proof is shorter and works under slightly less restrictive assumptions, which is why we include it in our first version of the paper.

Proposition 2.1. *Suppose that $\xi = (\xi_1, \dots, \xi_d)$ and $\eta = (\eta_1, \dots, \eta_d)$ are two locally admissible integrands for $X \in QV^d$. Then the pathwise Itô integrals $\int_0^t \xi(s) dX(s)$ and $\int_0^t \eta(s) dX(s)$ admit the continuous covariation*

$$\left[\int_0^\cdot \xi(s) dX(s), \int_0^\cdot \eta(s) dX(s) \right](t) = \sum_{i,j=1}^d \int_0^t \xi_i(s) \eta_j(s) d[X_i, X_j](s).$$

Proof. First, note that we can assume without loss of generality that $\xi = \eta$, by means of polarization. First step: Let F be a left-continuous non-anticipative functional of class $\mathbb{C}^{1,1}([0, T])$ such that $\mathcal{D}F, \nabla_X F \in \mathbb{B}$. We will show that $Y(t) := F(t, X^t, A^t)$ has the continuous quadratic variation

$$[Y](t) = \sum_{i,j=1}^d \int_0^t \partial_i F(s, X^s, A^s) \partial_j F(s, X^s, A^s) d[X_i, X_j](s).$$

First, observe that we can approximate the increments of F for the path (X, A) by the respective increments along the piecewise constant approximations (X^n, A^n) from (15), (16), due to the left-continuity of F . As above, denote $\delta X_s^n := X(s') - X(s)$, $s, s' \in \mathbb{T}_n$, and $s(u) := s$ such that $u \in [s, s']$. Then, a first-order Taylor expansion analogous to the one from the proof of Theorem 2.1 yields

$$\begin{aligned} F(s', X^{n,s'-}, A^{n,s'}) - F(s, X^{n,s-}, A^{n,s}) &= \int_{(s,s']} \mathcal{D}F(u, X^{n,s(u)}, A^{n,u-}) d\mu_n(u) \\ &\quad + \nabla_X F(s, X^{n,s-}, A^{n,s}) \delta X_s^n + r_s^n, \end{aligned}$$

where, for some $\theta \in [0, 1]$,

$$r_s^n := \left(\nabla_X F(s, X^{n,s-\theta\delta X_s^n}, A^{n,s}) - \nabla_X F(s, X^{n,s-}, A^{n,s}) \right) \delta X_s^n,$$

and hence $|r_s^n| \leq \epsilon_s^n |\delta X_s^n|$ with $\epsilon_s^n \rightarrow 0$. Thus,

$$\begin{aligned} &\sum_{s \in \mathbb{T}_n, s \leq t} \left(F(s', X^{n,s'-}, A^{n,s'}) - F(s, X^{n,s-}, A^{n,s}) \right)^2 \\ &= \sum_{s \in \mathbb{T}_n, s \leq t} \sum_{i,j=1}^d \partial_i F(s, X^{n,s-}, A^{n,s}) \partial_j F(s, X^{n,s-}, A^{n,s}) (X_i(s') - X_i(s)) (X_j(s') - X_j(s)) \\ &\quad + \sum_{s \in \mathbb{T}_n, s \leq t} \sum_{l,j=0}^m \int_{(s,s']} \mathcal{D}_l F(u, X^{n,s}, A^{n,u-}) d\mu_{n,l}(u) \int_{(s,s']} \mathcal{D}_j F(u, X^{n,s}, A^{n,u-}) d\mu_{n,j}(u) \\ &\quad + 2 \sum_{s \in \mathbb{T}_n, s \leq t} \int_{(s,s']} \mathcal{D}F(u, X^{n,s}, A^{n,u-}) \mu_n(du) \nabla_X F(s, X^{n,s-}, A^{n,s}) \delta X_s^n + \sum_{s \in \mathbb{T}_n, s \leq t} R_s^n. \end{aligned}$$

Since all appearing approximations have a d_∞ -distance from (X^s, A^s) less than η_n from (20), and $\mathcal{D}F, \nabla_X F \in \mathbb{B}$, we infer that

$$|R_s^n| \leq C\epsilon_s^n \left(|(X(s') - X(s)|^2 + \sum_{i=1}^d \sum_{l=0}^m |(X_i(s') - X_i(s)| |\mu_{n,l}(ds)| \right).$$

Moreover, $\mathcal{D}F(u, X^{n,s(u)}, A^{n,u-})$ and $\nabla_X F(s, X^{n,s-}, A^{n,s}) \mathbb{1}_{u \in (s, s']}$ are bounded, converge to $\mathcal{D}F(u, X^u, A^u)$ and $\nabla_X F(u, X^u, A^u)$, respectively, and all paths are left-continuous in u (see proof of Theorem 2.1). Thus we can use a “diagonal lemma” for vague convergence of measures in the form of [4, Lemma 12]. This gives us that the first term on the right-hand side of the above equation converges to

$$\sum_{i,j=1}^d \int_0^t \partial_i F(s, X^s, A^s) \partial_j F(s, X^s, A^s) d[X_i, X_j](s).$$

The second and third terms on the right-hand side of the above equation converge to 0, since μ corresponds to a continuous function whose components are of bounded variation, and hence all appearing covariations vanish (see [22, Remark 8]). The same argument gives that the error term converges to 0, since ϵ_s^n converges to 0.

Second step: By concentrating on small time intervals, we can assume without loss of generality that ξ is of the form $\xi(t) = \nabla_X F(t, X^t, A^t)$, $t \in [0, T]$, for A, F as in Definition 2.8. The change of variables formula, Theorem 2.1, thus implies

$$\begin{aligned} F(T, X^T, A^T) - F(0, X^0, A^0) &= \int_0^T \mathcal{D}F(s, X^s, A^s) \mu(ds) \\ &\quad - \frac{1}{2} \sum_{i,j=1}^d \int_0^T \nabla_{X,ij}^2 F(s, X^s, A^s) d[X]_{ij}(s) = \int_0^T \nabla_X F(s, X^s, A^s) dX(s). \end{aligned}$$

We introduce

$$B(t) := - \int_0^t \mathcal{D}F(s, X^s, A^s) \mu(ds) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \nabla_{X,ij}^2 F(s, X^s, A^s) d[X]_{ij}(s),$$

which belongs to the class $CBV([0, T])$, by standard properties of Stieltjes integrals. In particular, we have $[B] \equiv 0$. Moreover, setting $Y(t) := F(t, X^t, A^t) - F(0, X^0, A^0)$, we obtain

$$\left[\int_0^\cdot \nabla_X F(s, X^s, A^s) dX(s) \right](t) = [Y + B](t) = [Y](t).$$

Applying the first step yields

$$\begin{aligned} \left[\int_0^\cdot \xi(s) dX(s) \right](t) &= \left[\int_0^\cdot \nabla_X F(s, X^s, A^s) dX(s) \right](t) \\ &= [Y](t) = \sum_{i,j=1}^d \int_0^t \partial_i F(s, X^s, A^s) \partial_j F(s, X^s, A^s) d[X_i, X_j](s), \end{aligned}$$

which concludes the proof. \square

3 The associativity rule in functional pathwise Itô calculus

We are now ready to state and show the associativity property of the pathwise functional Itô integral (18). This extends [22, Theorem 13] to the functional setting. It turns out that associativity is crucial when discussing Itô differential equations in the pathwise setting (see [19, 22]). This fact will be illustrated for functional Itô differential equations in the subsequent section.

Let $X \in QV^d \cap \mathcal{U}^T$ and $\xi_{(1)}, \dots, \xi_{(\nu)}$ be locally admissible integrands for X . We define

$$Y_{(\ell)}(t) := \int_0^t \xi_{(\ell)}(s) dX(s), \quad \ell = 1, \dots, \nu. \quad (34)$$

Then, Proposition 2.1 implies that the continuous trajectory $Y = (Y_{(1)}, \dots, Y_{(\nu)})$ admits the continuous covariations

$$[Y_{(k)}, Y_{(\ell)}](t) = \sum_{i,j=1}^d \int_0^t \xi_{(k),i}(s) \xi_{(\ell),j}(s) d[X_i, X_j](s). \quad (35)$$

Theorem 3.1. *Let $X, \xi_{(1)}, \dots, \xi_{(\nu)}$ be as above and Y as in (34). Moreover, let $\eta = (\eta_1, \dots, \eta_\nu)$ be a locally admissible integrand for Y . Then $\sum_{\ell=1}^\nu \eta_\ell \xi_{(\ell)}$ is a locally admissible integrand for X and*

$$\int_0^T \eta(s) dY(s) = \int_0^T \sum_{\ell=1}^\nu \eta_\ell(s) \xi_{(\ell)}(s) dX(s). \quad (36)$$

For the proof we need the following auxiliary lemmas. The first one is a product rule for vertical derivatives, the second one a chain rule for both vertical and horizontal derivatives. Both extend statements from [14].

Lemma 3.1. *Let F, G be two non-anticipative vertically differentiable functionals such that $F, G, \nabla_X F, \nabla_X G \in \mathbb{F}_l^\infty$ and $F, G, \nabla_X F, \nabla_X G \in \mathbb{B}$. Then the product FG is again a non-anticipative vertically differentiable functional such that $FG, \nabla_X(FG) \in \mathbb{F}_l^\infty$ and $FG, \nabla_X(FG) \in \mathbb{B}$. Moreover,*

$$\partial_i(FG) = \partial_i F G + F \partial_i G \quad \text{for all } i = 1, \dots, d. \quad (37)$$

Proof. Let $X \in \mathcal{U}^T, A \in \mathcal{S}_{BV}^T$ in the following. For $i \in \{1, \dots, d\}$, we have

$$\begin{aligned} \partial_i(FG)(t, X^t, A^t) &= \lim_{h \rightarrow 0} \frac{(FG)(t, X^{t, h e_i}, A^t) - (FG)(t, X^t, A^t)}{h} \\ &= G(t, X^t, A^t) \partial_i F(t, X^t, A^t) + F(t, X^t, A^t) \partial_i G(t, X^t, A^t). \end{aligned}$$

Hence, FG is vertically differentiable with respect to X , since F and G are, and (37) holds.

Moreover, the gradient of FG is boundedness-preserving, since all functionals appearing on the right-hand side of (37) are, and since the product of two boundedness-preserving functionals is again boundedness-preserving. Analogously, all functionals appearing on the right-hand side of (37) are left-continuous, by our assumptions. Thus, both FG and its gradient, $\nabla_X(FG)$, are left-continuous, since it is easily checked that the product of two left-continuous (locally bounded) functionals is again left-continuous. This concludes the proof. \square

Lemma 3.2. Let $H : [0, T] \times \mathcal{V}^T \times \mathcal{S}_{BV}^T \mapsto \mathbb{R}$, where $V \subset \mathbb{R}^\nu$ open, $S \subset \mathbb{R}^n$ Borel, and $\tilde{F} : [0, T] \times \mathcal{U}^T \times \tilde{\mathcal{S}}_{BV}^T \mapsto \mathbb{R}^\nu$, where $\tilde{S} \subset \mathbb{R}^{\tilde{m}}$ Borel, be two non-anticipative functionals such that $H, \tilde{F} \in \mathbb{C}^{1,2}([0, T])$ and satisfy the regularity conditions from Theorem 2.1. Then, the composition $\tilde{H}(t, X, (D, \tilde{A})) := H(t, \tilde{F}^t(\cdot, X, \tilde{A}), D)$ defines a non-anticipative real-valued functional on $[0, T] \times \mathcal{U}^T \times \tilde{\mathcal{W}}_{BV}^T$, where $\tilde{W} \subset \mathbb{R}^{n+\tilde{m}}$ Borel, that is of class $\mathbb{C}^{1,2}([0, T])$ and satisfies the regularity conditions from Theorem 2.1. Moreover,

$$\partial_i \tilde{H} = \sum_{\ell=1}^{\nu} \partial_\ell H \partial_i \tilde{F}_{(\ell)}, \quad i = 1, \dots, d; \quad (38)$$

$$\nabla_{X,ij}^2 \tilde{H} = \sum_{\ell=1}^{\nu} \left(\sum_{m=1}^{\nu} \nabla_{Y,\ell m}^2 H \partial_j \tilde{F}_{(m)} \partial_i \tilde{F}_{(\ell)} + \partial_\ell H \nabla_{X,ij}^2 \tilde{F}_{(\ell)} \right), \quad i, j = 1, \dots, d; \quad (39)$$

$$\mathcal{D} \tilde{H} = \left(\mathcal{D}_0 H + \sum_{\ell=1}^{\nu} \partial_\ell H \mathcal{D}_0 \tilde{F}_{(\ell)}, \sum_{\ell=1}^{\nu} \partial_\ell H \mathcal{D}_1 \tilde{F}_{(\ell)}, \dots, \sum_{\ell=1}^{\nu} \partial_\ell H \mathcal{D}_{\tilde{m}} \tilde{F}_{(\ell)}, \mathcal{D}_1 H, \dots, \mathcal{D}_n H \right)^\top. \quad (40)$$

Proof. We can assume without loss of generality that $d = \nu = \tilde{m} = 1$. Since H is vertically differentiable with respect to Y , it follows that

$$H(t, Y^{t,h'}, D^t) = H(t, Y^t, D^t) + P(t, Y^t, D^t) h' + o(|h'|);$$

analogously, since \tilde{F} is vertically differentiable with respect to X , it follows that

$$\tilde{F}(t, X^{t,h}, \tilde{A}^t) = \tilde{F}(t, X^t, \tilde{A}^t) + Q(t, X^t, \tilde{A}^t) h + o(|h|).$$

This implies, with $h' = \tilde{F}(t, X^{t,h}, \tilde{A}^t) - \tilde{F}(t, X^t, \tilde{A}^t)$,

$$\begin{aligned} \tilde{H}(t, X^{t,h}, D^t, \tilde{A}^t) &= H(t, \tilde{F}^{t,h'}(\cdot, X, \tilde{A}), D^t) = H(t, Y^{t,h'}, D^t) \\ &= H(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t) + P(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t) (\tilde{F}(t, X^{t,h}, \tilde{A}^t) - \tilde{F}(t, X^t, \tilde{A}^t)) + o(|h'|) \\ &= H(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t) + P(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t) Q(t, X^t, \tilde{A}^t) h + o(|h|). \end{aligned}$$

Thus, \tilde{H} is vertically differentiable with respect to X , and its vertical derivative is given by

$$\partial_X \tilde{H}(t, X^t, D^t, \tilde{A}^t) = P(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t) Q(t, X^t, \tilde{A}^t) = \partial_Y H(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t) \partial_X \tilde{F}(t, X^t, \tilde{A}^t),$$

which shows (38). Since H and \tilde{F} are twice vertically differentiable with respect to Y and X , respectively, an application of the chain rule in the form of (38) and the product rule from the preceding lemma directly yields (39).

Now we turn to the proof of (40). Clearly, there is nothing to show for the last n components of the vector $\mathcal{D} \tilde{H}$. For the horizontal derivative with respect to \tilde{A} , we proceed as follows. Since \tilde{F} is horizontally differentiable with respect to \tilde{A} , it follows that

$$\tilde{F}(t, X^{t-h}, \tilde{A}^t) = \tilde{F}(t, X^{t-h}, \tilde{A}^{t-h}) + \Phi(t, X^t, \tilde{A}^t) \tilde{\mu}((t-h, t]) + o(|\tilde{\mu}((t-h, t])|).$$

This implies, with $h' = \tilde{F}(t, X^{t-h}, \tilde{A}^t) - \tilde{F}(t, X^{t-h}, \tilde{A}^{t-h})$,

$$\begin{aligned} \tilde{H}(t, X^{t-h}, D^{t-h}, \tilde{A}^t) &= H(t, \tilde{F}^{t-h,h'}(\cdot, X, \tilde{A}), D^{t-h}) = H(t, \tilde{F}^{t-h}(\cdot, X, \tilde{A}), D^{t-h}) \\ &+ P(t, \tilde{F}^{t-h}(\cdot, X, \tilde{A}), D^{t-h}) (\tilde{F}(t, X^{t-h}, \tilde{A}^t) - \tilde{F}(t, X^{t-h}, \tilde{A}^{t-h})) + o(|h'|) \\ &= H(t, \tilde{F}^{t-h}(\cdot, X, \tilde{A}), D^{t-h}) + P(t, \tilde{F}^{t-h}(\cdot, X, \tilde{A}), D^{t-h}) \Phi(t, X^t, \tilde{A}^t) \tilde{\mu}((t-h, t]) + o(|\tilde{\mu}((t-h, t])|). \end{aligned}$$

Thus, \tilde{H} is horizontally differentiable with respect to \tilde{A} with

$$\mathcal{D}\tilde{H}(t, X^t, D^t, \tilde{A}^t) = P(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t)\Phi(t, X^t, \tilde{A}^t) = \partial_Y H(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t)\mathcal{D}\tilde{F}(t, X^t, \tilde{A}^t),$$

which shows (40). Analogously, the statement follows for the first component of the vector $\mathcal{D}\tilde{H}$.

As observed in Lemma 3.1, products of boundedness-preserving (respectively, left-continuous) functionals are again boundedness-preserving (respectively, left-continuous). Hence, $\tilde{H} \in \mathbb{C}^{1,2}([0, T])$ and satisfies the regularity conditions from Theorem 2.1. \square

Proof of Theorem 3.1. As in the proof of Proposition 2.1, by concentrating on small intervals we can assume without loss of generality that $\xi_{(\ell)}$ is of the form $\xi_{(\ell)}(t) = \nabla_X F_{(\ell)}(t, X^t, A_{(\ell)}^t)$, $t \in [0, T]$, for $A_{(\ell)} \in \mathcal{S}_{CBV}^{\ell, T}$, where $S^\ell \subset \mathbb{R}^{m_\ell}$, and $F_{(\ell)}$ as in Definition 2.8. Then, Theorem 2.1 yields

$$\begin{aligned} F_{(\ell)}(T, X^T, A_{(\ell)}^T) - F_{(\ell)}(0, X^0, A_{(\ell)}^0) &= \int_0^T \mathcal{D}F_{(\ell)}(s, X^s, A_{(\ell)}^s) \mu_{(\ell)}(ds) \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^T \nabla_{X,ij}^2 F_{(\ell)}(s, X^s, A_{(\ell)}^s) d[X]_{ij}(s) + \int_0^T \nabla_X F_{(\ell)}(s, X^s, A_{(\ell)}^s) dX(s), \end{aligned}$$

where $\mu_{(\ell)}(ds) := (ds, A_{(\ell),1}(ds), \dots, A_{(\ell),m_\ell}(ds))^\top$. Introducing

$$A_{(\ell),m_\ell+1}(t) := \int_0^t \mathcal{D}F_{(\ell)}(s, X^s, A_{(\ell)}^s) \mu_{(\ell)}(ds) + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \nabla_{X,ij}^2 F_{(\ell)}(s, X^s, A_{(\ell)}^s) d[X]_{ij}(s) \quad (41)$$

and setting $\tilde{A}_{(\ell)} := (A_{(\ell),1}, \dots, A_{(\ell),m_\ell}, A_{(\ell),m_\ell+1})^\top$, we can write

$$Y_{(\ell)}(t) = F_{(\ell)}(t, X^t, A_{(\ell)}^t) - F_{(\ell)}(0, X^0, A_{(\ell)}^0) - A_{(\ell),m_\ell+1}(t) =: \tilde{F}_{(\ell)}(t, X^t, \tilde{A}_{(\ell)}^t). \quad (42)$$

Here, $\tilde{A}_{(\ell)}$ is a continuous function whose components are of bounded variation, by standard properties of Stieltjes integrals (e.g., [26, Theorem I.5 c]). Moreover, $\tilde{F}_{(\ell)}$ is a non-anticipative functional of class $\mathbb{C}^{1,2}([0, T])$ with $\nabla_X \tilde{F}_{(\ell)}(t, X, \tilde{A}_{(\ell)}) = \nabla_X F_{(\ell)}(t, X, A_{(\ell)})$, and the regularity conditions from Theorem 2.1 are satisfied for $F_{(\ell)}$ being replaced by $\tilde{F}_{(\ell)}$. Denoting

$$\tilde{F}(t, X, \tilde{A}) := \left(\tilde{F}_{(1)}(t, X, \tilde{A}_{(1)}), \dots, \tilde{F}_{(\nu)}(t, X, \tilde{A}_{(\nu)}) \right)^\top,$$

the identity (42) reads

$$Y(t) = \tilde{F}(t, X^t, \tilde{A}^t). \quad (43)$$

Again by concentrating on small intervals, η can be written without loss of generality as $\eta(t) = \nabla_Y H(t, Y^t, D^t)$, $t \in [0, T]$, for $D \in \mathcal{S}_{CBV}^T$, where $S \subset \mathbb{R}^m$, and H as in Definition 2.8. Using (43) as well as the notation $\nabla_X \tilde{F}(t, X, \tilde{A})$ for the matrix of vertical derivatives of \tilde{F} , Lemma 3.2 yields

$$\begin{aligned} \sum_{\ell=1}^\nu \eta_\ell(t) \xi_{(\ell)}(t) &= \sum_{\ell=1}^\nu \partial_\ell H(t, Y^t, D^t) \nabla_X \tilde{F}_{(\ell)}(t, X^t, \tilde{A}_{(\ell)}^t) \\ &= \nabla_Y H \left(t, \tilde{F}^t(\cdot, X, \tilde{A}), D^t \right) \cdot \nabla_X \tilde{F}(t, X^t, \tilde{A}^t) \\ &= \nabla_X \tilde{H}(t, X^t, \tilde{D}^t), \end{aligned}$$

where $\tilde{D} = (D, \tilde{A}) \in \mathcal{S}_{CBV}^T \times \tilde{\mathcal{S}}_{CBV}^T$, with $S \subset \mathbb{R}^m$, $\tilde{S} \subset \mathbb{R}^{\tilde{m}}$ Borel, and $\tilde{m} = \sum_{\ell=1}^{\nu} m_{\ell} + \nu$. Moreover,

$$\tilde{H}(t, X, (D, \tilde{A})) := H\left(t, \tilde{F}^t(\cdot, X, \tilde{A}), D\right)$$

defines a non-anticipative left-continuous functional that satisfies the requirements of Definition 2.8, also by Lemma 3.2. We hence infer that $\sum_{\ell=1}^{\nu} \eta_{\ell} \xi_{(\ell)}$ is admissible for X .

Using (43) we get

$$\begin{aligned} \int_0^T \eta(s) dY(s) &= \lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^{\nu} \partial_{\ell} H(s, Y^{n,s-}, D^{n,s})(F_{(\ell)}(s', X^{s'}, A_{(\ell)}^{s'}) - F_{(\ell)}(s, X^s, A_{(\ell)}^s)) \\ &\quad - \lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^{\nu} \partial_{\ell} H(s, Y^{n,s-}, D^{n,s})(A_{(\ell), m_{\ell}+1}(s') - A_{(\ell), m_{\ell}+1}(s)), \end{aligned} \quad (44)$$

where $Y^{n,s-}$ denotes the n -th approximation of Y , stopped at time $s-$. Since $\partial_{\ell} H(s, Y^{n,s-}, D^{n,s}) \mathbb{1}_{u \in (s, s']}$ is bounded and converges to $\eta_{\ell}(u)$, by left-continuity of $\nabla_Y H$, and both are left-continuous in u , by [4, Proposition 1], and since $A_{(\ell), m_{\ell}+1}$ is a continuous function of bounded variation, and hence corresponds to an atomless measure on $[0, T]$, we can use a “diagonal lemma” for vague convergence of measures in the form of [4, Lemma 12]. Thus, with the associativity of the Stieltjes integral (see, e.g., [26, Theorem I.6 b]) and the definition of $A_{(\ell), m_{\ell}+1}$, we infer for the second term in (44):

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^{\nu} \partial_{\ell} H(s, Y^{n,s-}, D^{n,s})(A_{(\ell), m_{\ell}+1}(s') - A_{(\ell), m_{\ell}+1}(s)) \\ &= \sum_{\ell=1}^{\nu} \int_0^T \eta_{\ell}(s) dF_{(\ell)}(s, X^s, A_{(\ell)}^s) \mu_{(\ell)}(ds) + \frac{1}{2} \sum_{\ell=1}^{\nu} \sum_{i,j=1}^d \int_0^T \eta_{\ell}(s) \nabla_{X,ij}^2 F_{(\ell)}(s, X^s, A_{(\ell)}^s) d[X]_{ij}(s). \end{aligned} \quad (45)$$

For the first term in (44), observe that, as in the proof of Proposition 2.1, we can approximate the increments of $F_{(\ell)}$ for the path $(X, A_{(\ell)})$ by the respective increments along the piecewise constant approximations $(X^n, A_{(\ell)}^n)$. As in the proof of Theorem 2.1, for $s \in \mathbb{T}_n$ we consider the following decomposition of these increments into “horizontal” and “vertical” terms:

$$\begin{aligned} F_{(\ell)}(s', X^{n,s'-}, A_{(\ell)}^{n,s'}) - F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) &= F_{(\ell)}(s', X^{n,s'-}, A_{(\ell)}^{n,s'}) - F_{(\ell)}(s, X^{n,s}, A_{(\ell)}^{n,s}) \\ &\quad + F_{(\ell)}(s, X^{n,s}, A_{(\ell)}^{n,s}) - F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}). \end{aligned} \quad (46)$$

For the first term on the right-hand side of (46), we have

$$F_{(\ell)}(s', X^{n,s'-}, A_{(\ell)}^{n,s'}) - F_{(\ell)}(s, X^{n,s}, A_{(\ell)}^{n,s}) = \int_{(0, s'-s]} dF_{(\ell)}(s+u, X^{n,s}, A_{(\ell)}^{n,s+u-}) \mu_{(\ell),n}(du), \quad (47)$$

where $\mu_{(\ell),n}$ is the measure corresponding to the n -th approximation $A_{(\ell)}^n$ of $A_{(\ell)}$. For the second term on the right-hand side of (46), we have

$$\begin{aligned} F_{(\ell)}(s, X^{n,s}, A_{(\ell)}^{n,s}) - F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) &= \nabla_X F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \delta X_s^n \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \nabla_{X,ij}^2 F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \delta X_{i,s}^n \delta X_{j,s}^n + r_{s,\ell}^n, \end{aligned} \quad (48)$$

where for some $\theta \in [0, 1]$,

$$r_{s,\ell}^n := \frac{1}{2} \sum_{i,j=1}^d \left(\nabla_{X,ij}^2 F_{(\ell)}(s, X^{n,s-}, \theta \delta X_s^n, A_{(\ell)}^{n,s}) - \nabla_{X,ij}^2 F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \right) \delta X_{i,s}^n \delta X_{j,s}^n.$$

Since $(X, A_{(\ell)})$, (Y, D) are continuous and hence, uniformly continuous on $[0, T]$, it follows that

$$\eta_{n,\ell} := \sup_{\alpha, \beta \in [0, T], |\alpha - \beta| \leq |\mathbb{T}_n|} \left(|A_{(\ell)}(\alpha) - A_{(\ell)}(\beta)| + |X(\alpha) - X(\beta)| + |D(\alpha) - D(\beta)| + |Y(\alpha) - Y(\beta)| \right)$$

converges to 0 as $n \rightarrow \infty$. Thus, since $\nabla_X^2 F_{(\ell)}$, $\mathcal{D}F_{(\ell)}$, and $\nabla_Y H$ are boundedness-preserving, we infer that for n large enough there exists a constant $C > 0$ such that

$$\begin{aligned} & \forall t < T, \forall (X', A'_{(\ell)}) \in \mathcal{U}^t \times \mathcal{S}_{BV}^{\ell,t}, \quad (Y', D') \in \mathcal{V}^t \times \mathcal{S}_{BV}^t, \\ & d_\infty \left((X^t, A_{(\ell)}^t), (X', A'_{(\ell)}) \right) + d_\infty \left((Y^t, D^t), (Y', D') \right) < \eta_{n,\ell} \\ & \Rightarrow |\partial_\ell H(t, Y', D') \mathcal{D}F_{(\ell)}(t, X', A'_{(\ell)})| \leq C, \quad \|\partial_\ell H(t, Y', D') \nabla_X^2 F_{(\ell)}(t, X', A'_{(\ell)})\| \leq C. \end{aligned}$$

We now sum over $s \in \mathbb{T}_n$ and $\ell = 1, \dots, \nu$. As in the proof of Theorem 2.1, we have that

$$\sum_{s \in \mathbb{T}_n} \left(F_{(\ell)}(s', X^{n,s'-}, A_{(\ell)}^{n,s'}) - F_{(\ell)}(s, X^{n,s}, A_{(\ell)}^{n,s}) \right) = \int_{(0,T]} \mathcal{D}F_{(\ell)} \left(u, X^{n,s(u)}, A_{(\ell)}^{n,u-} \right) \mu_{\ell,n}(du)$$

converges to $B_{(\ell)}(T) := \int_0^T \mathcal{D}F_{(\ell)}(s, X^s, A_{(\ell)}^s) \mu_{(\ell)}(ds)$, using a “diagonal lemma” for vague convergence of measures in the form of [4, Lemma 12]. The quantity $\partial_\ell H(s, Y^{n,s-}, D^{n,s}) \mathbb{1}_{u \in (s, s']}$ is bounded, converges to $\eta_\ell(u)$, by left-continuity of $\nabla_Y H$, and both are left-continuous in u , by [4, Proposition 1]. Moreover, since $B_{(\ell)}$ as a function in t is continuous and of bounded variation, and hence corresponds to an atomless measure on $[0, T]$, another application of [4, Lemma 12], in conjunction with the associativity of the Stieltjes integral (see [26, Theorem I.6 b]), yields that

$$\begin{aligned} & \sum_{\ell=1}^\nu \sum_{s \in \mathbb{T}_n} \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \left(F_{(\ell)}(s', X^{n,s'-}, A_{(\ell)}^{n,s'}) - F_{(\ell)}(s, X^{n,s}, A_{(\ell)}^{n,s}) \right) \\ & \rightarrow \sum_{\ell=1}^\nu \int_0^T \eta_\ell(s) \mathcal{D}F_{(\ell)}(s, X^s, A_{(\ell)}^s) \mu_{(\ell)}(ds). \end{aligned} \tag{49}$$

Moreover, using (48) we obtain that

$$\begin{aligned} & \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^\nu \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \left(F_{(\ell)}(s, X^{n,s}, A_{(\ell)}^{n,s}) - F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \right) \\ & = \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^\nu \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \nabla_X F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \cdot \delta X_s^n \\ & \quad + \frac{1}{2} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^\nu \sum_{i,j=1}^d \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \nabla_X^2 F_{(\ell),ij}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \delta X_{i,s}^n \delta X_{j,s}^n \\ & \quad + \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^\nu \partial_\ell H(s, Y^{n,s-}, D^{n,s}) r_{s,\ell}^n. \end{aligned} \tag{50}$$

The quantity $\partial_\ell H(s, Y^{n,s-}, D^{n,s}) \nabla_X^2 F_{(\ell), ij}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \mathbb{1}_{u \in (s, s']}$ is bounded by C , converges to $\eta_\ell(u) \nabla_X^2 F_{(\ell), ij}(u, X^u, A_{(\ell)}^u)$, by left-continuity of $\nabla_Y H$, $\nabla_X^2 F_{(\ell)}$, and both are left-continuous in u (by [4, Proposition 1]). Since $X \in QV^d$, an application of [4, Lemma 12] gives that

$$\begin{aligned} & \frac{1}{2} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^{\nu} \sum_{i,j=1}^d \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \nabla_X^2 F_{(\ell), ij}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \delta X_{i,s}^n \delta X_{j,s}^n \\ & \rightarrow \frac{1}{2} \sum_{\ell=1}^{\nu} \sum_{i,j=1}^d \int_0^T \eta_\ell(u) \nabla_X^2 F_{(\ell), ij}(u, X^u, A_{(\ell)}^u) d[X]_{ij}(u) \end{aligned} \quad (51)$$

as $n \rightarrow \infty$. Using the same argument, since $|\partial_\ell H(s, Y^{n,s-}, D^{n,s}) r_{s,\ell}^n|$ is bounded by $\epsilon_{s,\ell}^n |\delta X_s^n|^2$, where $\epsilon_{s,\ell}^n$ converges to 0, the remainder term, $\sum_{\ell=1}^{\nu} \sum_{s \in \mathbb{T}_n} \partial_\ell H(s, Y^{n,s-}, D^{n,s}) r_{s,\ell}^n$, converges to 0.

Applying (49) and (51) we obtain for the first term in (44):

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^{\nu} \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \left(F_{(\ell)}(s', X^{s'}, A_{(\ell)}^{s'}) - F_{(\ell)}(s, X^s, A_{(\ell)}^s) \right) \\ & = \lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^{\nu} \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \left(F_{(\ell)}(s', X^{n,s'-}, A_{(\ell)}^{n,s'}) - F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \right) \\ & = \sum_{\ell=1}^{\nu} \int_0^T \eta_\ell(s) \mathcal{D}F_{(\ell)}(s, X^s, A_{(\ell)}^s) \mu_{(\ell)}(ds) + \lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^{\nu} \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \nabla_X F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \delta X_s^n \\ & \quad + \frac{1}{2} \sum_{\ell=1}^{\nu} \sum_{i,j=1}^d \int_0^T \eta_\ell(s) \nabla_X^2 F_{(\ell), ij}(u, X^s, A_{(\ell)}^s) d[X]_{ij}(s). \end{aligned} \quad (52)$$

Thus, we receive in conjunction with (45),

$$\int_0^T \eta(s) dY(s) = \lim_{n \rightarrow \infty} \sum_{s \in \mathbb{T}_n} \sum_{\ell=1}^{\nu} \partial_\ell H(s, Y^{n,s-}, D^{n,s}) \nabla_X F_{(\ell)}(s, X^{n,s-}, A_{(\ell)}^{n,s}) \cdot \delta X_s^n.$$

Since we have already established in the first step that $\sum_{\ell=1}^{\nu} \eta_\ell(s) \xi_{(\ell)}(s)$ is admissible for X , this concludes the proof. \square

4 Applications to Itô Differential Equations

The associativity rule derived in the preceding section guarantees that informal computations with Itô differentials typically lead to correct statements. It is therefore of fundamental importance and crucial for many applications. For instance, the elementary associativity rule from [22] was derived for a pathwise treatment of CPPI strategies and it allows to transfer the Doss–Sussmann method to pathwise Itô calculus (see Section 2.3 in [19]). Our original motivation for establishing the associativity rule within functional Itô calculus stems from stochastic portfolio theory; see our companion paper [24]. To illustrate already in our present paper why the associativity rule is such a crucial property, we will now use it to prove existence and uniqueness results for pathwise linear Itô differential equations whose coefficients are non-anticipative functionals.

Let $X \in QV^d$, $Y \in QV$ be such that all covariations $[Y, X_i]$, $i = 1, \dots, d$, exist and are continuous, and $(\sigma_1, \dots, \sigma_d)$ be an \mathbb{R}^d -valued non-anticipative functional on the Skorohod space $D([0, T], \mathbb{R})$. Then, $Z \in C([0, T], \mathbb{R})$ is called a solution of the *linear Itô differential equation*

$$dZ(t) = dY(t) + Z(t)\sigma(t) dX(t), \quad (53)$$

with initial condition $Z(0) = z$, if the mapping $t \mapsto Z(t)\sigma(t)$ is a locally admissible integrand for X and if Z satisfies the integral form of (53): $Z(t) = z + \int_0^t Z(s)\sigma(s) dX(s)$, $0 \leq t \leq T$.

Theorem 4.1 (Existence and uniqueness of the homogeneous linear IDE). *Suppose that σ is a locally admissible integrand for $X \in QV^d$. Then, for any $z \in \mathbb{R}$, the homogeneous linear Itô differential equation*

$$dZ(t) = Z(t)\sigma(t) dX(t), \quad (54)$$

with initial condition $Z(0) = z$, has the unique solution

$$Z(t) = z \exp \left(\int_0^t \sigma(s) dX(s) - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \sigma_i(s)\sigma_j(s) d[X]_{ij}(s) \right). \quad (55)$$

The solution Z of the above equation is the *Doléans-Dade* exponential of

$$I(t) := \int_0^t \sigma(s) dX(s), \quad (56)$$

and will be denoted by $\mathcal{E}(I)(t)$ in the following.

Proof of Theorem 4.1. Let us first show that Z satisfies the required equation. Since I from (56) has the continuous quadratic variation $[I](t) = \sum_{i,j=1}^d \int_0^t \sigma_i(s)\sigma_j(s) d[X]_{ij}(s)$, by Proposition 2.1, setting

$$K(t) := I(t) - \frac{1}{2}[I](t) = I(t) - \sum_{i,j=1}^d \int_0^t \sigma_i(s)\sigma_j(s) d[X]_{ij}(s),$$

and applying Föllmer's pathwise Itô formula [15] to K and $f(k) = e^k$ we obtain:

$$Z(t) = f(K(t)) = Z(0) + \int_0^t Z(s) dK(s) + \frac{1}{2} \int_0^t Z(s) d[K](s) = z + \int_0^t Z(s)\sigma(s) dX(s).$$

Here we have used Theorem 3.1, with $\eta = Z$ and $\xi = \sigma$, the associativity of the Stieltjes integral from [26, Theorem I.6 b], and the fact that $[K](t) = [I](t)$, due to [22, Remark 8].

Let us now turn to the proof of the uniqueness of solutions. Throughout the proof we let Z be as defined in (55) and suppose that \tilde{Z} is another solution of (54) with initial condition z .

I. $z \geq 0$: Since $Z > 0$ in this case, applying the pathwise Itô formula [15] to the function $f(z, \tilde{z}) = \tilde{z}/z$ and the paths Z, \tilde{Z} yields that

$$\frac{\tilde{Z}(t)}{Z(t)} = 1 + \int_0^t \begin{pmatrix} 1/Z(s) \\ -\tilde{Z}(s)/Z^2(s) \end{pmatrix} d \begin{pmatrix} \tilde{Z}(s) \\ Z(s) \end{pmatrix} + \int_0^t \frac{\tilde{Z}(s)}{Z^3(s)} d[Z](s) - \int_0^t \frac{1}{Z^2(s)} d[Z, \tilde{Z}](s). \quad (57)$$

Since both Z and \tilde{Z} satisfy the integral form of (54), the associativity result from Theorem 3.1 gives for $\nu = 2$, $\eta = \begin{pmatrix} 1/Z \\ -\tilde{Z}/Z^2 \end{pmatrix}$, $\xi_{(1)}(s) = \tilde{Z}(s)\sigma(s)$, and $\xi_{(2)}(s) = Z(s)\sigma(s)$ that

$$\int_0^t \begin{pmatrix} 1/Z(s) \\ -\tilde{Z}(s)/Z^2(s) \end{pmatrix} d \begin{pmatrix} \tilde{Z}(s) \\ Z(s) \end{pmatrix} = \int_0^t \left(\frac{1}{Z(s)} \tilde{Z}(s) - \frac{\tilde{Z}(s)}{Z^2(s)} Z(s) \right) \sigma(s) dX(s) = 0,$$

and so the pathwise Itô integral vanishes. Moreover, using Proposition 2.1 for the quadratic variation $[Z]$ and covariation $[Z, \tilde{Z}]$ we obtain, in conjunction with the associativity of the Stieltjes integral from [26, Theorem I.6 b],

$$\int_0^t \frac{\tilde{Z}(s)}{Z^3(s)} d[Z](s) = \sum_{i,j=1}^d \int_0^t \sigma_i(s) \sigma_j(s) \frac{\tilde{Z}(s)}{Z(s)} d[X]_{ij}(s) = \int_0^t \frac{1}{Z^2(s)} d[Z, \tilde{Z}](s).$$

Plugging these results back into (57) we arrive at $\frac{\tilde{Z}(t)}{Z(t)} \equiv 1$, which is the desired uniqueness in case $z > 0$.

II. $z < 0$: If \tilde{Z} is any solution of (54) with $\tilde{Z}(0) = z$, then $\hat{Z} := -\tilde{Z}$ also is a solution of (54) with initial condition $\hat{Z}(0) = -z > 0$. Hence, uniqueness holds by the previous argument.

III. $z = 0$: Suppose by way of contradiction that there exists a non-zero solution \tilde{Z} of (54) with $\tilde{Z}(0) = 0$. Then, the stopping time $\tau_n := \inf \{t > 0 \mid \tilde{Z}(t) = \frac{1}{n}\}$ will be finite for sufficiently large $n \in \mathbb{N}$. Introducing the time-shifted paths $\hat{Z}(t) := \tilde{Z}(t + \tau_n)$, $\hat{\sigma}(t) := \sigma(t + \tau_n)$, and $\hat{X}(t) := X(t + \tau_n)$, we have for sufficiently large n ,

$$\hat{Z}(t) = \frac{1}{n} + \int_0^t \hat{Z}(s) \hat{\sigma}(s) d\hat{X}(s), \quad t \geq 0,$$

which is equivalent to $\hat{Z}(t) = \frac{1}{n} \mathcal{E} \left(\int_0^t \hat{\sigma}(s) d\hat{X}(s) \right) (t)$, by previous arguments. It follows that

$$\tilde{Z}(t + \tau_n) = \frac{1}{n} \exp \left(\int_{\tau_n}^{t+\tau_n} \sigma(s) dX(s) - \frac{1}{2} \sum_{i,j=1}^d \int_{\tau_n}^{t+\tau_n} \sigma_i(s) \sigma_j(s) d[X_i, X_j](s) \right), \quad t \geq 0. \quad (58)$$

Clearly, the integrals inside the exponential function are uniformly bounded in n . Thus, letting n go to infinity in (58) yields $\tilde{Z}(t) = 0$ for all $t \geq \lim_n \tau_n$, which is the desired contradiction. Hence, $Z(t) = 0$ is the only solution with initial value $Z(0) = 0$. \square

For the general *inhomogeneous* linear Itô differential equation (53) the following result extends the standard “variation of constants” technique.

Theorem 4.2 (Existence and uniqueness of the inhomogeneous linear IDE). *Suppose that σ is a locally admissible integrand for $X \in QV^d$ and $Y \in QV$ is such that all covariations $[Y, X_i]$, $i = 1, \dots, d$, exist and are continuous. Then, for $Z^0(t) := \mathcal{E} \left(\int_0^t \sigma(s) dX(s) \right) (t)$, and any $z \in \mathbb{R}$, the inhomogeneous linear Itô differential equation*

$$dZ(t) = dY(t) + Z(t)\sigma(t) dX(t), \quad (59)$$

with initial condition $Z(0) = z$, has the unique solution

$$Z(t) = Z^0(t) \left(z + \int_0^t \frac{1}{Z^0(s)} dY(s) - \int_0^t \frac{1}{Z^0(s)} d \left[Y, \int_0^\cdot \sigma(s) dX(s) \right] (s) \right). \quad (60)$$

Proof. If Z and \tilde{Z} are solutions of (59) with the same initial condition, then their difference $\hat{Z} := Z - \tilde{Z}$ satisfies $d\hat{Z}(t) = \hat{Z}(t)\sigma(t) dX(t)$ with initial condition $\hat{Z}(0) = 0$. Hence, Theorem 4.1 implies $\hat{Z} \equiv 0$, which shows the uniqueness of solutions. In the second step, an application of Föllmer's pathwise product rule and of the associativity result from Theorem 3.1 shows that Z indeed solves (59). \square

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